Chapter 14: Partial Derivatives

Section 1:

Definition 1:

R is a region in xy-plane. (x_0, y_0) a point in the xy-plane.

- If there is a disc of center (x_0, y_0) which lies entirely in R, we say that (x_0, y_0) is an interior point of R.
- If any disc of center (x_0, y_0) intersect both R and the complement of R, we say that (x_0, y_0) is a boundary point of R.
- The set of all interior points of R is called the interior of R.
- The set of all boundary points of R is called the boundary of R.
- If each point of R is an interior point, we say that R is an open region. (In this case, R is the same as its interior).
- If R contains its boundary, we say that R is a closed region.
- If R lies inside a disc of fixed radius, we say that R is a bounded region.

Definition 2:

The function of 2 variables is a function whose domain is a region in the xy-plane and whose range is a subset of the set IR.

Definition 3:

$f(x, y)$ is a function of 2 variables. c is in the range of f. The set of all points (x, y, z) in space such that $z = f(x, y)$ is called the graph of f. The graph of f is also called the surface $z = f(x, y)$. The set of all points (x, y) in the plane such that $f(x, y) = c$ is called a level curve of f.

Definition 4:

$f(x, y, z)$ is a function of 3 variables. Suppose c in Range f.
The set of all points (x, y, z) in space such that $f(x, y, z) = c$ is called the level surface of f.

Section 2:

Definition 1:

Suppose R is a region. The point (x_0, y_0) point in the plane. If (x_0, y_0) is either an interior point of R or a boundary point of R, we say that (x_0, y_0) is a limit point of R.

Definition 2:

$f(x, y)$ is a function of 2 variables.
(x_0, y_0) is a limit point of Domain f.
We say $f(x, y)$ has a limit L as (x, y) approaches to (x_0, y_0) and write $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if to any given $\varepsilon > 0$, there corresponds an $\delta > 0$ such that: $\left.\begin{array}{l} 0 < \text{distance from } (x_0, y_0) \text{ to } (x, y) < \delta \\ (x, y) \in \text{Domain } f \end{array}\right\} \Rightarrow |f(x, y) - L| < \varepsilon$

i.e.: $\left.\begin{array}{l} 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \\ (x, y) \in \text{Domain } f \end{array}\right\} \Rightarrow |f(x, y) - L| < \varepsilon$

Theorem 1:

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} x = x_0$

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} y = y_0$

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} k = k$

Theorem 2:

$g(x,y), f(x,y)$ are two functions of two variables. (x_0, y_0) is a limit point of Domain *f* and Domain *g*. Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = L_2$ Then:

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) \pm g(x,y) = L_1 \pm L_2$

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)g(x,y) = L_1 L_2$

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L_1}{L_2}$ provided that both $g(x,y)$ and L_2 are different from zero.

Definition 3:

Suppose $f(x,y)$ is a function of two variables.

Suppose (x_0, y_0) is a limit point of Domain *f*.

We say that *f* is continuous at (x_0, y_0) if :

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$

(ii) (x_0, y_0) is actually in Domain *f*.

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$

Section 3:

Definition 1:

Suppose $f(x,y)$ is a function of 2 variables.

(x_0, y_0) is an interior point of Domain *f*.

For (x,y) in Domain *f* we define $\Delta x = x - x_0$ and $\Delta y = y - y_0$

$\Delta f = f(x,y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y)$

The partial derivatives of *f* at (x_0, y_0) are defined as:
$$\begin{cases} \dfrac{\partial f}{\partial x}(x_0, y_0) = \lim\limits_{\substack{\Delta x\to 0 \\ \Delta y = 0}} \dfrac{\Delta f}{\Delta x} = \lim\limits_{\substack{\Delta x\to 0 \\ \Delta y = 0}} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ \dfrac{\partial f}{\partial y}(x_0, y_0) = \lim\limits_{\substack{\Delta y\to 0 \\ \Delta x = 0}} \dfrac{\Delta f}{\Delta y} = \lim\limits_{\substack{\Delta x\to 0 \\ \Delta y = 0}} \dfrac{f(x, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \end{cases}$$

Provided that these two limit exist.

Definition 2:

Suppose $f(x,y)$ is a function of 2 variables.

(x_0, y_0) is an interior point of Domain *f*. Then:

We say that *f* is differentiable at (x_0, y_0) if:

(i) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ both exist at (x_0, y_0).

(ii) $\Delta f = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial x}(x_0 y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ where $\varepsilon_0 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

Section 4:

Theorem 1: Chain Rule:

$f(x, y, z)$ is a differentiable function of 3 variables.

$g(t), h(t),$ and $\ell(t)$ differentiable functions of one variable.

If $x = g(t)$, $y = h(t)$, $z = \ell(t)$, and $w = f(x, y, z)$

Then: w is a differentiable function of one variable and : $\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$

Corollary To Chain Rule:

$f(x, y, z)$ is a differentiable function of 3 variables.

$g(r, s), h(r, s),$ and $\ell(r, s)$ differentiable functions of two variable.

If $x = g(r, s)$, $y = h(r, s)$, $z = \ell(r, s)$, and $w = f(x, y, z)$

Then: w is a differentiable function of two variable and :

$$\frac{dw}{dr} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} + f_z \frac{\partial z}{\partial r}$$
$$\frac{dw}{ds} = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s} + f_z \frac{\partial z}{\partial s}$$

Theorem 2:

$F(x, y)$ differentiable function of 2 variables:

Suppose the equation $F(x, y) = 0$ defines y implicitly as a function of x. Say $y = h(x)$ then: $\frac{dy}{dx} = -\frac{F_x}{F_y}$

Provided that $F_y \neq 0$

Section 5:

Definition 1:

$f(x, y, z)$ differentiable function of 3 variables.

Suppose $u = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a unit vector in space (i.e. $\sqrt{a^2 + b^2 + c^2} = 1$)

(x_0, y_0, z_0) is an interior point of Domain f.

L is the line through (x_0, y_0, z_0) parallel to **u**.

For (x, y, z) in L, define Δs as the directed distance from (x_0, y_0, z_0) to (x, y, z).

The directed derivation of f at (x_0, y_0, z_0) in the direction of **u** is defined as: $D_u f(x_0, y_0, z_0) = \lim_{\Delta s \to 0} \frac{\Delta f}{\Delta s}$ provided that this limit exists.

Formula for the Directed Derivation :

$D_u f = \mathbf{u}.\nabla f(x_0, y_0, z_0)$ where $\nabla f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\mathbf{i} + f_y(x_0, y_0, z_0)\mathbf{j} + f_z(x_0, y_0, z_0)\mathbf{k}$

Some Important Observations:

1. f increases most rapidly at (x_0, y_0) in the direction of $\nabla f(x_0, y_0)$. The value of the derivation in this direction is $|\nabla f(x_0, y_0)|$.

2. f decreases most rapidly at (x_0, y_0) in the direction of $-\nabla f(x_0, y_0)$. The value of the derivation in this direction is $-|\nabla f(x_0, y_0)|$.

Section 7:

Definition 1:

$f(x, y)$ a differentiable function of 2 variables.

(a,b) interior point of Domain *f*.

If $f_x(a,b) = f_y(a,b) = 0$ then we say that the point (a,b) is a critical point of *f*.

Theorem 1:

$f(x, y)$ a differentiable function of 2 variables.

(a,b) interior point of Domain *f*.

Then: *f* has a local max or a local min at $(a,b) \Rightarrow (a,b)$ is a critical point of *f*.

Theorem 2: (Second Derivation Test):

$f(x, y)$ a differentiable function of 2 variables.

(a,b) is a critical point of *f*.

Then:

(i) $\left.\begin{array}{l} f_{xx} f_{yy} - (f_{xy})^2 > 0 \text{ at } (a,b) \\ f_{xx} > 0 \end{array}\right\} \Rightarrow f$ has a local min at (a,b).

(ii) $\left.\begin{array}{l} f_{xx} f_{yy} - (f_{xy})^2 > 0 \text{ at } (\mathrm{a,b}) \\ f_{xx} < 0 \end{array}\right\} \Rightarrow f$ has a local max at (a,b).

(iii) $f_{xx} f_{yy} - (f_{xy})^2 < 0$ at $(\mathrm{a,b}) \Rightarrow f$ has a saddle point at (a,b).

(iv) $f_{xx} f_{yy} - (f_{xy})^2 = 0$ at $(\mathrm{a,b}) \Rightarrow$ Test fails.

Theorem 3:

$f(x, y)$ a function of 2 variables.

R is a region in the plane.

Then: $\left.\begin{array}{l} f \text{ continuous everywhere in } R \\ R \text{ is closed and bounded} \end{array}\right\} \Rightarrow f$ has an absolute max and an absolute min in *R*.

Section 8:

Theorem 1: Lagrange Multipliers:

$f(x, y, z)$, $g(x, y, z)$ are 2 differentiable functions of 3 variables.

Let *S* be the level surface $g(x, y, z) = 0$ and suppose that *S* is contained in the Domain *f*.

Then: if a point $(x_0, y_0, z_0) \in S$ is a point where *f* has a local min or local max (as a function of *S*), then:

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$